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ON INTERMEDIATE ORBITS.

By Dr. G. W. Hill, West Nyack, N. Y.

The assumption of the Keplerian ellipse as the first approximation to the motion of a planet leads to some inconveniences, the worst of which, perhaps, is that the mean longitude at the epoch suffers a perturbation proportional to the time, and thus the mean motion in longitude is not the same in the perturbed and unperturbed orbits. In the case of Saturn this perturbation amounts to 110″ per year. This inconvenience would be avoided if, developing the perturbative function in a series proceeding according to the cosines of multiples of the angle H contained at the sun by the radii of the planets thus

$$R = \varphi_0(r, r') + \varphi_1(r, r') \cos H + \varphi_2(r, r') \cos 2H + \ldots,$$

we should suppose $\varphi_0(r, r')$ annexed to the potential of the attraction exerted by the sun and the resulting differential equations integrated and the expressions of the coordinates thus deduced regarded as the first approximation. Then the following approximations could be obtained by the method of variation of the elements or otherwise by attributing to R the value

$$R = \varphi_1(r, r') \cos H + \varphi_2(r, r') \cos 2H + \ldots.$$

In this way of approaching the question, the first approximation to the motion of a system of planets revolving about their central body would involve a potential function containing the radii of the planets alone, without their longitudes or latitudes ; thus the force acting on each planet would be directed along its radius, but would be a function of all the radii. The orbits are then all plane curves whose planes pass through the central body, and thus we know precisely what functions the latitudes and the reduced longitudes are of the orbit longitudes. Also there is equable description of areas by the radii, and when the latter are known functions of the time the determination of the orbit longitudes is reduced to quadratures. Thus the question is narrowed to the finding of the radii as functions of the time.

Let $r, r', \ldots v, v', \ldots$ denote the radii and orbit longitudes of a system of planets, $m, m', \ldots$ their masses and M the mass of the central body ; $\mu = M + m$, $\mu' = M + m', \ldots$; $R, R', \ldots$ the perturbative functions for $m, m', \ldots$. Then

the differential equations for determining the radii are

$$\frac{d^2r}{dt^2} - r\frac{dv^2}{dt^2} + \frac{\mu}{r^2} = \frac{\partial R}{\partial r},$$

$$\frac{d^2r'}{dt^2} - r'\frac{dv'^2}{dt^2} + \frac{\mu'}{r'^2} = \frac{\partial R'}{\partial r'},$$

. . . . . .

But, h, h', ... being constants, we have

$$\frac{dv}{dt} = \frac{h}{r^2}, \quad \frac{dv'}{dt} = \frac{h'}{r'^2}, \ \ldots.$$

The functions R, R', ... are to be reduced to

$$R = m'\varphi(r, r') + m''\varphi(r, r'') + \ldots,$$

$$R' = m\varphi(r', r) + m''\varphi(r', r'') + \ldots,$$

. . . . . .

where we have the relation $\varphi(x, y) = \varphi(y, x)$. To evaluate the function φ, let the symbol $\mathfrak{M}$ denote the operation of taking the arithmetico-geometrical mean, then

$$\varphi(r, r') = \frac{1}{\mathfrak{M}(r' + r, r' - r)} = \frac{1}{\mathfrak{M}(r', \sqrt{r'^2 - r^2})},$$

where it is supposed that $r' > r$. The two quantities under the symbol $\mathfrak{M}$ may be brought still nearer to equality by continuing the operation, and writing $\sqrt{r'} = \alpha$, $\sqrt[4]{r'^2 - r^2} = \beta$, we have

$$\varphi(r, r') = \frac{1}{\mathfrak{M}[\frac{1}{2}(\alpha^2 + \beta^2), \alpha\beta]} = \frac{1}{\mathfrak{M}[\frac{1}{4}(\alpha+\beta)^2, \sqrt{\frac{1}{2}\alpha\beta(\alpha^2 + \beta^2)}]}.$$

And if r and r' are not too near each other a sufficiently approximate value is

$$\varphi(r, r') = \frac{2\sqrt{2}}{(\alpha + \beta)\sqrt[4]{\alpha\beta(\alpha^2 + \beta^2)}}.$$

If we write

$$\Omega = \frac{\mu m}{r} + \frac{\mu' m'}{r'} + \ldots - \frac{1}{2}\frac{mh^2}{r^2} - \frac{1}{2}\frac{m'h'^2}{r'^2} - \ldots$$
$$+ mm'\varphi(r, r') + mm''\varphi(r, r'') + m'm''\varphi(r', r'') + \ldots,$$

the differential equations determining r, r', ... become

$$m\frac{d^2r}{dt^2} = \frac{\partial\Omega}{\partial r}, \quad m'\frac{d^2r'}{dt^2} = \frac{\partial\Omega}{\partial r'}, \ \ldots.$$

From which we derive the integral equation

$$m\frac{dr^2}{dt^2} + m'\frac{dr'^2}{dt^2} + \ldots = 2(\Omega + C),$$

C being the arbitrary constant. The left member of this cannot be a negative quantity. Consequently, if we construct in a space of n dimensions, n being the number of planets, the surface whose equation is

$$\Omega + C = 0,$$

the representative point P, whose coordinates are the values of the variables $r, r', \ldots$, must lie on the positive side of this surface. In such a system of planets as our solar system composed of the eight major planets, at least one fold of this surface is closed ; and it is within this that the representative point P always falls. Thus, in this case, we are able to set definite inferior and superior limits to $r, r', \ldots$, which, however, in general are not the *minima-minimorum* or *maxima-maximorum* values of the variables, since the point P only attains the surface when all the planets are together on the lines of their apsides.

A particular solution of the foregoing system of differential equations can be obtained in the following way : Put

$$\frac{\partial\Omega}{\partial r} = 0, \quad \frac{\partial\Omega}{\partial r'} = 0, \ \ldots,$$

and solve these equations, regarding $r, r', \ldots$ as the unknowns. Let

$$r = a, \quad r' = a', \ \ldots$$

be a system of values satisfying them. Then the last equations evidently satisfy the differential equations. When this process is applied to the solar system the values found for the a do not greatly differ from the mean distances of the several planets from the sun. The excessive smallness of the m relatively to M brings it about that there is but one real solution ; and quite approximately, by neglecting in Ω all the terms dependent on the functions φ, we get

$$a = \frac{h^2}{\mu}, \quad a' = \frac{h'^2}{\mu'}, \ \ldots.$$

By substituting these values in the terms arising from these functions we can arrive at more exact values for the a ; and this process can be repeated as often as is deemed necessary. Evidently all this comes to the adjustment of

the planets at such distances from the sun that for the given values of the m and the h the centripetal force may be equal to the centrifugal force in each case, thus rendering it possible that all the orbits may be circles.

Desiring now to find inferior and superior limits to the particular variable $r^{(i)}$, in the surface $\Omega + C = 0$ we suppose this variable to be a function of all the rest, and thus that when it arrives at a maximum or minimum value we have

$$\frac{\partial r^{(i)}}{\partial r} = 0\,, \quad \frac{\partial r^{(i)}}{\partial r'} = 0\,, \ \ldots\,,$$

except that $\frac{\partial r^{(i)}}{\partial r^{(i)}} = 1$. In order to find limits for the values of $r^{(i)}$ we take the group of equations

$$\frac{\partial \Omega}{\partial r} = 0\,, \quad \frac{\partial \Omega}{\partial r'} = 0\,, \ \ldots\,,$$

and in it replace the equation $\frac{\partial \Omega}{\partial r^{(i)}} = 0$ by the equation $\Omega + C = 0$. Solving these equations, regarding r, r', ... as the unknowns, we shall always, in cases like our solar system, be able to find two sets of corresponding real values for these quantities such that the two values of $r^{(i)}$ shall contain between them the observed value of this radius from which the value of C was derived. And no other roots can be found which satisfy this condition. In ordinary cases these roots are non-multiple and there is but one solution to the problem.

For the sake of illustration, limiting ourselves to two planets, we may take a hypothetical case, suggested by Jupiter and Saturn. Assume as the values of the planetary masses

$$m = \frac{1}{1047.355}\,, \quad m' = \frac{1}{3501.6}\,.$$

From Leverrier's Tables for the time 1875, Jan. 1.0, we get

$$\log r = 0.7367630\,, \quad \log r' = 0.9953504\,;$$

and, with the Julian year as the unit of time,

$$\log \frac{dr}{dt} = 8.0841634\, n\,, \quad \log \frac{dr'}{dt} = 8.8776073\,.$$

From the same source the values of the osculating elements necessary for the computations are

$$\log a = 0.7162505,\ \log e = 8.6865018,\ \log a' = 0.9794334,\ \log e' = 8.7291462.$$

Accommodated to the units of length and time we employ,

$$\log\mu = 1.5967563,\ \log\mu' = 1.5964658,\ \log m = 8.5762479,\ \log m' = 8.0520753.$$

$$\log h^2 = 2.3119809,\ \log h'^2 = 2.5746499.$$

When these numerical values are substituted in the equation $\Omega + C = 0$, it is found that $C = -0.166513246$. Employing brackets to denote numbers corresponding to common logarithms, the equation $\Omega + C = 0$ becomes in this case

$$[0.1730042]\, r^{-1} - [0.5871988]\, r^{-2} + [9.6485411]\, r'^{-1} - [0.3256952]\, r'^{-2} + [6.6283232]\, r^{-1}\, \zeta\left(\frac{r}{r'}\right) = 0.166513246\,.$$

And the equations $\frac{\partial\Omega}{\partial r} = 0$, $\frac{\partial\Omega}{\partial r'} = 0$ become

$$2\,[0.5871988]\, r^{-1} - [0.1730042] + [6.6283232]\, \frac{r^2}{r'^2}\, \zeta'\left(\frac{r}{r'}\right) = 0\,,$$

$$2\,[0.3256952]\, r'^{-2} - [9.6485411] - [6.6283232]\left[\zeta\left(\frac{r}{r'}\right) + \frac{r}{r'}\,\zeta'\left(\frac{r}{r'}\right)\right] = 0\,,$$

where ζ denotes the same function as Laplace's $b_{\frac{1}{2}}^{(0)}$ is of α.

The solution of the first and third of these equations gives inferior and superior limits for r, the solution of the first and second the same for r'. Employing the tentative process we obtain as the corresponding values of the logarithms of the radii in the four solutions

$$\log r = 0.6926819\,,\quad 0.7390339\,,\quad 0.7152497\,,\quad 0.7152334\,;$$

$$\log r' = 0.9776575\,,\quad 0.9776183\,,\quad 0.9238410\,,\quad 1.0390527\,.$$

The solution of the second and third equations gives the values of the radii for which the planets could describe uniformly circles. These are

$$\log a = 0.7152396\,,\quad \log a' = 0.9776403\,.$$

If we assume that the maximum eccentricities of Jupiter and Saturn are respectively 0.0594902 and 0.0845677, the minimum and maximum values of $\log r$ and $\log r'$ are severally

$$\log r = 0.6895907,\ \log r = 0.7413343,\ \log r' = 0.9411223,\ \log r' = 1.0147527.$$

The differences between the former values and these are, of course, due to the neglect of the terms of R involving the angle H.

For the purpose of graphically exhibiting the numerical results just obtained we may suppose a system of rectangular axes to be drawn with r and r' as coordinates. The four sets of corresponding values of these variables give four points on the oval which constitutes one of the branches of the curve having $\Omega + C = 0$ as its equation. These suffice to roughly indicate the course of the oval within which the representative point P, exhibiting the simultaneous values of the radii, always keeps. The position for which both radii are constant is well towards the centre of the oval. If we draw the right line bisecting the right angle at the origin formed by the axes of coordinates, the intersection of this line or its nonintersection with the oval will show whether there is intrusion or not of the spheres of the planets on each other. In the case here treated there is no intrusion; each planet maintains its character as exterior or interior; consequently the distance between them can never vanish, and stability of motion is assured.

The given example shows that there must exist a large domain in which it is possible to expand the function Ω in a series of powers and products of the deviations of the radii from their mean values. Here we will limit the exposition to the case of two planets, as the formulæ given can be readily extended when there are more. Let us put

$$r = a + \frac{x}{\sqrt{m}}, \quad r' = a' + \frac{x'}{\sqrt{m'}},$$

where the second terms are small relatively to the first. The differential equations then become

$$\frac{d^2x}{dt^2} = \frac{\partial \Omega}{\partial x}, \quad \frac{d^2x'}{dt^2} = \frac{\partial \Omega}{\partial x'}.$$

In developing Ω in a series of ascending powers and products of x and x' we will stop with terms of three dimensions. The constant term can be omitted as its retention would only serve to modify the value of the arbitrary constant C. As a and a' are corresponding roots of the equations $\frac{\partial \Omega}{\partial r} = 0, \frac{\partial \Omega}{\partial r'} = 0$, the terms of one dimension vanish. Also the term in xx' can be supposed absent, for, if present, it could be removed by a linear and orthogonal transformation of variables. We write therefore

$$\Omega = -\tfrac{1}{2} a^2x^2 - \tfrac{1}{2} a'^2x'^2 + \tfrac{1}{3} cx^3 + bx^2x' + b'xx'^2 + \tfrac{1}{3} c'x'^3,$$

where a, a', b, b', c, c' are constants.*

* The reader is asked not to confound these a and a' with the quantities previously denoted by the same letters.

Adopting two constants n and n' at present left indeterminate, but to be determined hereafter so as to fulfil certain conditions, the differential equations can be written

$$\frac{d^2x}{dt^2} + n^2x = (n^2 - a^2)x + cx^2 + 2bxx' + b'x'^2,$$

$$\frac{d^2x'}{dt^2} + n'^2x' = (n'^2 - a'^2)x' + c'x'^2 + 2b'xx' + bx^2.$$

Before integrating these it will be well to ascertain superior and inferior limits for x and x'. Those for x are determined by the equations

$$C - \tfrac{1}{2}a^2x^2 - \tfrac{1}{2}a'^2x'^2 + \tfrac{1}{3}cx^3 + bx^2x' + b'xx'^2 + \tfrac{1}{3}c'x'^3 = 0,$$

$$-a'^2x' + c'x'^2 + 2b'xx' + bx^2 = 0.$$

If we put $\eta = \pm\frac{\sqrt{2C}}{a}$, this will be a small quantity of the same order as x and the limiting values of x can be developed in an infinite series of powers of η. Thus

$$x = \eta + \frac{1}{3}\frac{c}{a^2}\eta^2 + \left[\frac{1}{2}\frac{b^2}{a^2a'^2} + \frac{5}{18}\frac{c^2}{a^4}\right]\eta^3 + \left[\frac{b^2b'}{a^2a'^4} + \frac{b^2c}{a^4a'^2} + \frac{8}{27}\frac{c^3}{a^6}\right]\eta^4 + \ldots,$$

where we take η positively for the superior limit and negatively for the inferior. The corresponding value of x' can be derived from the equation

$$x' = \frac{b}{a'^2}x^2 + 2\frac{bb'}{a'^4}x^3 + \frac{b^2c'}{a'^6}x^4 + \ldots.$$

The limits for x', by putting $\eta = \pm\frac{\sqrt{2C}}{a'}$, can be obtained from those of x by removing the accent from the symbols which there have it and applying it to those which are destitute of it.

By neglecting the right-hand members of the differential equations as last written their integrals are simply

$$x = e\cos\varphi, \quad x' = e'\cos\varphi',$$

e and e' being arbitrary constants and φ and φ' are arguments increasing proportionably to t so that $\frac{d\varphi}{dt} = n$, $\frac{d\varphi'}{dt} = n'$. By substituting these expressions for x and x',

$$cx^2 + 2bxx' + b'x'^2 = \tfrac{1}{2}(ce^2 + b'e'^2) + \tfrac{1}{2}ce^2\cos 2\varphi + bee'\cos(\varphi + \varphi') + bee'\cos(\varphi - \varphi') + \tfrac{1}{2}b'e'^2\cos 2\varphi',$$

$$c'x'^2 + 2b'xx' + bx^2 = \tfrac{1}{2}(c'e'^2 + be^2) + \tfrac{1}{2}c'e'^2\cos 2\varphi' + b'ee'\cos(\varphi+\varphi') + b'ee'\cos(\varphi-\varphi') + \tfrac{1}{2}be^2\cos 2\varphi\,.$$

Whence it follows that x contains the additional terms of the second order with respect to e and e'

$$x = \frac{1}{2n^2}(ce^2 + b'e'^2) - \frac{1}{6n^2}ce^2\cos 2\varphi + \frac{bee'}{n^2-(n+n')^2}\cos(\varphi+\varphi') + \frac{bee'}{n^2-(n-n')^2}\cos(\varphi-\varphi') + \frac{1}{2}\frac{b'e'^2}{n^2-4n'^2}\cos 2\varphi'\,,$$

and x' terms which are obtained from those of x by simply interchanging the accents.

Pushing the approximation to terms of the third order, x^2 is found to contain the following terms of this order:

$$x^2 = [\tfrac{5}{6}ce^2 + b'e'^2]\frac{e}{n^2}\cos\varphi + \frac{2be^2e'}{4n^2-n'^2}\cos\varphi' - \frac{1}{6}\frac{ce^3}{n^2}\cos 3\varphi + \frac{be^2e'}{n^2-(n+n')^2}\cos(2\varphi+\varphi') + \frac{be^2e'}{n^2-(n-n')^2}\cos(2\varphi-\varphi') + \frac{1}{2}\frac{b'ee'^2}{n^2-4n'^2}\cos(\varphi+2\varphi') + \frac{1}{2}\frac{b'ee'^2}{n^2-4n'^2}\cos(\varphi-2\varphi')\,.$$

The similar terms of x'^2 are obtained from those of x^2 by interchanging the accents. The terms of the third order of xx' are

$$xx' = \left[\frac{1}{2}\frac{c'e'^2+be^2}{n'^2} + \frac{b(e'^2-\frac{1}{4}e^2)}{4n^2-n'^2}\right]e\cos\varphi + \left[\frac{1}{2}\frac{ce^2+b'e'^2}{n^2} + \frac{b'(e^2-\frac{1}{4}e'^2)}{4n'^2-n^2}\right]e'\cos\varphi' + \frac{1}{4}\frac{be^3}{n'^2-4n^2}\cos 3\varphi + \frac{1}{4}\frac{b'e'^3}{n^2-4n'^2}\cos 3\varphi' - \left[\frac{b'}{2n'+n} + \frac{c}{6n}\right]\frac{e^2e'}{2n}\cos(2\varphi+\varphi') + \left[\frac{b'}{2n'-n} - \frac{c}{6n}\right]\frac{e^2e'}{2n}\cos(2\varphi-\varphi') - \left[\frac{b}{2n+n'} + \frac{c'}{6n'}\right]\frac{ee'^2}{2n'}\cos(\varphi+2\varphi') + \left[\frac{b}{2n-n'} - \frac{c'}{6n'}\right]\frac{ee'^2}{2n'}\cos(\varphi-2\varphi').$$

The coefficients $n^2 - a^2$ and $n'^2 - a'^2$ are of the second order, hence limiting ourselves to terms of the third order

$$(n^2-a^2)x = (n^2-a^2)e\cos\varphi\,,\quad (n'^2-a'^2)x' = (n'^2-a'^2)e'\cos\varphi'\,.$$

The constants e and e' being arbitrary, the coefficients of $\cos\varphi$ in x and of $\cos\varphi'$ in x' need receive no corrections in the following approximations. Hence the coefficient of $\cos\varphi$ in the right member of the first differential equation should vanish, as also that of $\cos\varphi'$ in the second. This gives us the two equations

$$n^2 - a^2 + \left[\tfrac{5}{6}\frac{c^2}{n^2} + \frac{b^2}{n'^2} - \tfrac{1}{2}\frac{b^2}{4n^2 - n'^2}\right]e^2 + \left[\frac{b'c}{n^2} + \frac{bc'}{n'^2} + \frac{2b^2}{4n^2 - n'^2} + \frac{2b'^2}{4n'^2 - n^2}\right]e'^2 = 0\,,$$

$$n'^2 - a'^2 + \left[\tfrac{5}{6}\frac{c'^2}{n'^2} + \frac{b'^2}{n^2} - \tfrac{1}{2}\frac{b'^2}{4n'^2 - n^2}\right]e'^2 + \left[\frac{b'c}{n^2} + \frac{bc'}{n'^2} + \frac{2b^2}{4n^2 - n'^2} + \frac{2b'^2}{4n'^2 - n^2}\right]e^2 = 0\,,$$

of which the second is obtained from the first by interchange of accents. To the degree of approximation to which they are pushed they serve to determine the values of n and n' as functions of the constants a, a', b, b', c, c', e, e'. As the next terms which would appear in these equations through the following approximations are of the fourth order with reference to e and e', the errors of n and n', as determined from these equations, are of the same order.

The remaining terms of the third order in the right member of the first differential equation are

$$\left[\tfrac{1}{2}\frac{b^2}{n'^2 - 4n^2} - \tfrac{1}{6}\frac{c^2}{n^2}\right]e^3\cos 3\varphi + \left[\tfrac{1}{2}\frac{bb'}{n^2 - 4n'^2} - \tfrac{1}{6}\frac{b'c'}{n'^2}\right]e'^3\cos 3\varphi'$$

$$-\left[\frac{bc}{2nn' + n'^2} + \tfrac{1}{6}\frac{bc}{n^2} + \frac{bb'}{2nn' + n^2} + \tfrac{1}{2}\frac{bb'}{4n^2 - n'^2}\right]e^2e'\cos(2\varphi + \varphi')$$

$$+\left[\frac{bc}{2nn' - n'^2} - \tfrac{1}{6}\frac{bc}{n^2} + \frac{bb'}{2nn' - n^2} - \tfrac{1}{2}\frac{bb'}{4n^2 - n'^2}\right]e^2e'\cos(2\varphi - \varphi')$$

$$-\left[\tfrac{1}{2}\frac{b'c}{4n'^2 - n^2} + \tfrac{1}{6}\frac{bc'}{n'^2} + \frac{b^2}{2nn' + n'^2} + \frac{b'^2}{2nn' + n^2}\right]ee'^2\cos(\varphi + 2\varphi')$$

$$-\left[\tfrac{1}{2}\frac{b'c}{4n'^2 - n^2} + \tfrac{1}{6}\frac{bc'}{n'^2} - \frac{b^2}{2nn' - n'^2} - \frac{b'^2}{2nn' - n^2}\right]ee'^2\cos(\varphi - 2\varphi')\,.$$

The similar terms of the right member of the second differential equation are

obtained by interchange of accents in the preceding expression. The terms of the third order in x are then

$$\left[\frac{c^2}{n^2}+\frac{3b^2}{4n^2-n'^2}\right]\frac{e^3}{48n^2}\cos 3\varphi+\tfrac{1}{6}\left[\frac{b'c'}{n'^2}+\frac{3bb'}{4n'^2-n^2}\right]\frac{e'^3}{9n'^2-n^2}\cos 3\varphi'$$

$$+\left[\frac{bc}{2nn'+n'^2}+\tfrac{1}{6}\frac{bc}{n^2}+\frac{bb'}{2nn'+n^2}+\tfrac{1}{2}\frac{bb'}{4n^2-n'^2}\right]\frac{e^2e'}{3n^2+4nn'+n'^2}\cos(2\varphi+\varphi')$$

$$-\left[\frac{bc}{2nn'-n'^2}-\tfrac{1}{6}\frac{bc}{n^2}+\frac{bb'}{2nn'-n^2}-\tfrac{1}{2}\frac{bb'}{4n^2-n'^2}\right]\frac{e^2e'}{3n^2-4nn'+n'^2}\cos(2\varphi-\varphi')$$

$$+\left[\tfrac{1}{2}\frac{b'c}{4n'^2-n^2}+\tfrac{1}{6}\frac{bc'}{n'^2}+\frac{b^2}{2nn'+n'^2}+\frac{b'^2}{2nn'+n^2}\right]\frac{ee'^2}{4nn'+4n'^2}\cos(\varphi+2\varphi')$$

$$-\left[\tfrac{1}{2}\frac{b'c}{4n'^2-n^2}+\tfrac{1}{6}\frac{bc'}{n'^2}-\frac{b^2}{2nn'-n'^2}-\frac{b'^2}{2nn'-n^2}\right]\frac{ee'^2}{4nn'-4n'^2}\cos(\varphi-2\varphi').$$

The similar terms in x' are obtained from these by interchange of accents.

On many occasions it will be interesting to know the equivalent of C, the arbitrary constant attached to the integral equation, in terms of $a, a', b, b', c, c', e, e'$. To obtain this we compute the non-periodic terms of x^2, $\frac{dx^2}{dt^2}$, x^3, x^2x'. The non-periodic terms of x'^2, $\frac{dx'^2}{dt^2}$, x'^3, x'^2x can be derived from these by interchange of accents. Limiting ourselves to these terms we have

$$x^2=\tfrac{1}{2}e^2+\frac{19}{72}\frac{c^2}{n^4}e^4+\left[\frac{4n^2+n'^2}{n'^2(4n^2-n'^2)^2}b^2+\tfrac{1}{2}\frac{b'c'}{n'^4}\right]e^2e'^2+\left[\tfrac{1}{8}\frac{b'^2}{(4n'^2-n^2)^2}+\frac{b'^2}{4n'^4}\right]e'^4,$$

$$\frac{dx^2}{dt^2}=\tfrac{1}{2}n^2c^2+\frac{c^2}{18n^2}e^4+\frac{4n^4-3n^2n'^2+n'^4}{n'^2(4n^2-n'^2)^2}b^2e^2e'^2+\tfrac{1}{2}\frac{n'^2}{(4n'^2-n^2)^2}b'^2e'^4,$$

$$x^3=\frac{5}{8n^2}ce^4+\frac{3}{4n^2}b'e^2e'^2,$$

$$x^2x'=\tfrac{1}{8}\frac{8n^2-3n'^2}{n'^2(4n^2-n'^2)}be^4+\left[\frac{b}{4n^2-n'^2}+\frac{c'}{4n'^2}\right]c^2e'^2.$$

Substituting these values in the integral equation and eliminating n^2 and n'^2 by means of the two equations which determine them, we find

$$C=\tfrac{1}{2}(a^2e^2+a'^2e'^2)-\left[\frac{96a^4-68a^2a'^2+9a'^4}{16a'^2(4a^2-a'^2)^2}b^2+\frac{37}{144}\frac{c^2}{a^2}\right]e^4$$

$$+\left[\frac{8a^4-18a^2a'^2+5a'^4}{2a'^2(4a^2-a'^2)^2}b^2+\frac{8a'^4-18a^2a'^2+5a^4}{2a^2(4a'^2-a^2)^2}b'^2-\tfrac{3}{4}\frac{b'c}{a^2}-\tfrac{3}{4}\frac{bc'}{a'^2}\right]e^2e'^2$$

$$-\left[\frac{96a'^4-68a^2a'^2+9a^4}{16a^2(4a'^2-a^2)^2}b'^2+\frac{37}{144}\frac{c'^2}{a'^2}\right]e'^4.$$

It is necessary to notice the effect of the vanishing of the divisors introduced by the integration. In the terms of the second order if $n = 2n'$ the coefficient of $\cos 2\varphi'$ in x and the coefficient of $\cos(\varphi - \varphi')$ in x' apparently become infinite. Also, if $n' = 2n$, the coefficient of $\cos(\varphi - \varphi')$ in x and the coefficient of $\cos 2\varphi$ in x' are in like case. This evidently is the analytical warning that in these cases the arguments 2φ, $2\varphi'$, $\varphi - \varphi'$ cannot be distinguished from φ and φ'. Thus the coefficients of their cosines should have some indetermination. But let us see what conditions the quantities a, a', b, b', c, c', e, e' must satisfy that either of these relations may have place. We shall not allow that any of the quantities a, a', n, n' can be infinite. Then, from the equations determining n and n', it is plain that, in the first case, either $b' = 0$, or $2e^2 - e'^2 = 0$, and in the second case, either $b = 0$, or $e^2 - 2e'^2 = 0$. Thus all the mentioned coefficients in x and x', instead of becoming infinite in these cases, really take the form $\frac{0}{0}$.

The particular supposition of $e = 0$, $e' = 0$ makes the first approximation vanish, and it is plain from the differential equations that $x = 0$, $x' = 0$ form a particular solution of them in which n and n' are quite indeterminate. This case therefore does not demand further consideration. But suppose that, in the first case, $b' = 0$ or, in the second, $b = 0$. Then, in Ω, the term $b'xx'^2$ or the term bx^2x' disappears, and the terms in x and x' having indeterminate coefficients should also disappear. However, the relation $n = 2n'$ or $n' = 2n$, without being exactly fulfilled, may be so very nearly. Let us take the first. If, in the equations determining n and n' we make $n^2 = 4n'^2$ and $n'^2 = a'^2$ except in the first terms and those which involve the divisor $4n'^2 - n^2$, we shall have quadratic equations for the determination of n^2 and n'^2. Thus, putting

$$a'^2 = a'^2 - (\tfrac{5}{6}\, c'^2 + \tfrac{1}{4}\, b'^2)\,\frac{e'^2}{a'^2} - (\tfrac{1}{4}\, b'c + bc' + \tfrac{2}{15}\, b^2)\,\frac{e^2}{a'^2},$$

$$\beta = \tfrac{1}{4}\, a^2 - a'^2 + (\tfrac{5}{6}\, c'^2 + \tfrac{1}{4}\, b'^2 - \tfrac{1}{16}\, b'c - \tfrac{1}{4}\, bc' - \tfrac{1}{30}\, b^2)\,\frac{e'^2}{a'^2}$$
$$+ (\tfrac{1}{4}\, b'c + bc' - \tfrac{13}{120}\, b^2 - \tfrac{5}{96}\, c^2)\,\frac{e^2}{a'^2}$$

the solution of these quadratic equations will give

$$n^2 = 4a'^2 \pm 4\sqrt{\beta^2 - b'^2(2e^2 - e'^2)},$$

$$n'^2 = a'^2 - \tfrac{1}{2}\,\beta \pm \tfrac{1}{2}\sqrt{\beta^2 - b'^2(2e^2 - e'^2)},$$

$$4n'^2 - n^2 = -2\beta \mp 2\sqrt{\beta^2 - b'^2(2e^2 - e'^2)}.$$

The ambiguity of the sign before the radical is determined from the following considerations. When β is a somewhat large positive quantity it is evident

that we ought to take the upper sign, as always approximately $n'^2 = a'^2$, and then $n^2 = 4a'^2 + 4\beta$, thus n^2 is larger than $4n'^2$. Hence, if we begin with n^2 larger than $4n'^2$ and keep diminishing β, continuity demands that the upper sign be maintained. And we may persist in diminishing β until, if $2e^2 - e'^2$ is a positive quantity, $\beta = +\sqrt{b'^2(2e^2 - e'^2)}$ when $n'^2 = a'^2 - \frac{1}{2}\sqrt{b'^2(2e^2 - e'^2)}$ and $n^2 = 4a'^2$; thus n^2 is still larger than $4n'^2$. But if β is a somewhat large negative quantity it is evident that the lower sign must be attributed to the radical; and then approximately $n'^2 = a'^2$, $n^2 = 4a'^2 + 4\beta$; that is, here n^2 is less than $4n'^2$. Then β may be numerically diminished until $\beta = -\sqrt{b'^2(2e^2 - e'^2)}$, when $n'^2 = a'^2 + \frac{1}{2}\sqrt{b'^2(2e^2 - e'^2)}$ and $n^2 = 4a'^2$; and n^2 is still less than $4n'^2$. On the other hand if $2e^2 - e'^2$ is a negative quantity, β in both cases can be numerically diminished until it vanishes, and then, in the first case, we have $n'^2 = a'^2 + \frac{1}{2}\sqrt{b'^2(e'^2 - 2e^2)}$, $n^2 = 4a'^2 + 4\sqrt{b'^2(e'^2 - 2e^2)}$, and, in the second case, $n'^2 = a'^2 - \frac{1}{2}\sqrt{b'^2(e'^2 - 2e^2)}$, $n^2 = 4a'^2 - 4\sqrt{b'^2(e'^2 - 2e^2)}$.

From all this it is apparent that we cannot have $4n'^2 - n^2 = 0$ unless two conditions are satisfied. The first is $\beta = 0$, and the second either $b' = 0$ or $2e^2 - e'^2 = 0$. If the second condition is not fulfilled the limits towards which tend the coefficients $\frac{1}{2}\dfrac{b'e'^2}{n^2 - 4n'^2}$ and $\dfrac{b'ee'}{n'^2 - (n - n')^2}$, severally belonging to $\cos 2\varphi'$ in x and $\cos(\varphi - \varphi')$ in x', can be written so as to cover all the cases

$$\pm\frac{1}{4}\sqrt{\frac{e'^2}{\pm(2e^2 - e'^2)}} \quad \text{and} \quad \pm\sqrt{\frac{ee'}{\pm(2e^2 - e'^2)}}.$$

The ambiguous sign within the radical must be taken so as to render the quantity following positive; that without the radical depends at once on the sign of b' and on whether β has been supposed to approach the limit from positive or negative values. These limits are independent of b', and it will be perceived that instead of being of the second order with respect to e and e' they are of the same order as the first terms of x and x'. The second supposition of the second condition viz. $2e^2 - e'^2 = 0$ renders these limits infinite; but it is probable that this conclusion would be modified if terms of the fourth order were included in the equations determining n and n'.

There is no need of giving the similar investigation for the assumption $n' = 2n$. All the results of the latter are obtained from those of the former assumption by simply interchanging the accents. Thus the limits of the coefficient of $\cos 2\varphi$ in x' and the coefficient of $\cos(\varphi - \varphi')$ in x are severally

$$\pm\frac{1}{4}\sqrt{\frac{e^2}{\pm(2e'^2 - e^2)}} \quad \text{and} \quad \pm\sqrt{\frac{ee'}{\pm(2e'^2 - e^2)}}.$$

In all this we have assumed that the quantity under the radical sign in the expressions for n^2 and n'^2 must not be negative. This is simply to insure stability of motion. If the constants of the problem are such as to make it negative, or if either n^2 and n'^2 come out negative, the problem has still a real solution but the motion is unstable.

The question of periodic solutions is intimately connected with the vanishing of integrating divisors. If we suppose that $in + i'n' = 0$, where i and i' are integers prime to each other, by adopting an argument φ' such that $\frac{d\varphi}{dt} = i' \frac{d\psi}{dt}$ and $\frac{d\varphi'}{dt} = -i \frac{d\psi}{dt}$, the series for x and x' reduce to the form

$$x \text{ or } x' = a_0 + a_1 \cos \varphi' + \beta_1 \sin \varphi' + a_2 \cos 2\varphi' + \beta_2 \sin 2\varphi' + \ldots .$$

The variables then return to the same values after φ' has gone through a circumference as also do their differential coefficients. If we conceive a system of rectangular coordinates for graphically exhibiting the values of x and x', the representative point P, in its motion, will describe a certain curve which, if the variables are confined within limits, will lie within a limited portion of the plane. This curve may have multiple points ; let us see what is the condition necessary and sufficient that this may have place. The motion of x and x' being a constant oscillation between minimum and maximum values, the curve must be either a spiral having no multiple points, in which case there would be a progressive diminution or augmentation of the minimum and maximum values of x and x', or there must be an infinite number of multiple points. The periodic solution is the case which lies between these two classes of solutions. We may conceive the angle of intersection at the multiple point to constantly diminish through variation of the arbitrary constants of the problem until it vanishes, when the periodic solution is reached.

Let us suppose that the solution of the system of differential equations

$$\frac{d^2x}{dt^2} = \frac{\partial \Omega}{\partial x}, \qquad \frac{d^2x'}{dt^2} = \frac{\partial \Omega}{\partial x'},$$

admitting the integral

$$\frac{1}{2}\frac{dx^2 + dx'^2}{dt^2} = \Omega + C,$$

has been discovered and that it may be written

$$x = \mathrm{f}_1(t), \quad x' = \mathrm{f}_2(t).$$

Let t' denote another value of the independent variable t ; then, if the equations

$$\mathrm{f}_1(t') = \mathrm{f}_1(t), \quad \mathrm{f}_2(t') = \mathrm{f}_2(t),$$

t and t' being regarded as the unknowns, are satisfied by real values of t and t', the curve under discussion will have multiple points; but, if the roots of these equations are all imaginary, the curve will be a spiral. It appears that when x and x' are contained within finite limits the roots t and t' are all real and infinite in number; and thus that, in general, there are an infinite number of multiple points.

Let us now suppose that the arbitrary constants are adjusted so that, in addition to the equations $\mathrm{f}_1(t') = \mathrm{f}_1(t)$, $\mathrm{f}_2(t') = \mathrm{f}_2(t)$ being satisfied by certain values of t and t', the equation

$$\frac{\mathrm{f}_2'(t')}{\mathrm{f}_1'(t')} = \frac{\mathrm{f}_2'(t)}{\mathrm{f}_1'(t)}$$

is satisfied by the same values. (Accents have been used to denote differentiation of the form of f.) Then it is evident that the last equation is the condition of the presence of a periodic solution; and if we put $t' - t = P$, the equations

$$\mathrm{f}_1(t + P) = \mathrm{f}_1(t), \quad \mathrm{f}_2(t + P) = \mathrm{f}_2(t),$$

are satisfied for all values of t.

A periodic solution must have at least one arbitrary constant less than in the general case. But, for the differential equations we are treating, there is usually a periodic solution in which there are only two arbitrary constants. This happens when both variables arrive simultaneously at their maximum or minimum values. Integrating the differential equations by the application of Maclaurin's Theorem, we should have, in general,

$$x = a + bt + \frac{\partial \Omega}{\partial x}\frac{t^2}{2} + \ldots,$$

$$x' = a' + b't + \frac{\partial \Omega}{\partial x'}\frac{t^2}{2} + \ldots,$$

where a, a', b, b', are the arbitrary constants, and in the derivatives of Ω it is understood that x and x' are to be replaced severally by a and a'. In the case where both variables arrive simultaneously at a maximum or minimum value, and we count t from this epoch, not only $b = 0$, $b' = 0$, but also all the coefficients of the odd powers of t in x and x', as is easy to see from the successive differentiation of the derivatives of Ω with respect to the time. We may therefore put

$$x = a + a_1 t^2 + a_2 t^4 + a_3 t^6 + \ldots, = \mathrm{f}_1(t),$$

$$x' = a' + a'_1 t^2 + a'_2 t^4 + a'_3 t^6 + \ldots. = \mathrm{f}_2(t).$$

In the previous equations for determining the times of the representative point P passing through multiple points, we may suppose $t' = -t$, and the equations $\mathfrak{f}_1(-t) = \mathfrak{f}_1(t)$, $\mathfrak{f}_2(-t) = \mathfrak{f}_2(t)$ are fulfilled for all values of t. Also the equation showing the presence of a periodic solution is, in this case, an identity. This is explained by the circumstance that the point P having attained the limiting curve whose equation is $\Omega + C = 0$ returns on its previous path. Hence the condition necessary for the existence of a periodic solution in this case is that when the path is sufficiently prolonged backward it should again meet the limiting curve for real velocities. The motion of P will then be a swinging back and forth on this arc. This is tantamount to saying that the values of a and a' must be so adjusted that a value t_1 for t can be found which will make x and x' have such values that they satisfy the equation $\Omega + C = 0$.

It may be of interest to know the earlier terms of the expansions of x and x' in powers of t. For the sake of brevity we put

$$2\Psi = \left[\frac{\partial \Omega}{\partial x}\right]^2 + \left[\frac{\partial \Omega}{\partial x'}\right]^2.$$

Then

$$x = a + \frac{\partial \Omega}{\partial x}\frac{t^2}{1.2} + \frac{\partial \Psi}{\partial x}\frac{t^3}{1.2.3.4} + \left\{3\left[\frac{\partial \Omega}{\partial x}\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial \Omega}{\partial x'}\frac{\partial^2 \Psi}{\partial x \partial x'}\right] - 2\left[\frac{\partial^2 \Omega}{\partial x^2}\frac{\partial \Psi}{\partial x} + \frac{\partial^2 \Omega}{\partial x \partial x'}\frac{\partial \Psi}{\partial x'}\right]\right\}\frac{t^5}{6!} + \cdots$$

$$x' = a' + \frac{\partial \Omega}{\partial x'}\frac{t^2}{1.2} + \frac{\partial \Psi}{\partial x'}\frac{t^3}{1.2.3.4} + \left\{3\left[\frac{\partial \Omega}{\partial x'}\frac{\partial^2 \Psi}{\partial x'^2} + \frac{\partial \Omega}{\partial x}\frac{\partial^2 \Psi}{\partial x \partial x'}\right] - 2\left[\frac{\partial^2 \Omega}{\partial x'^2}\frac{\partial \Psi}{\partial x'} + \frac{\partial^2 \Omega}{\partial x \partial x'}\frac{\partial \Psi}{\partial x}\right]\right\}\frac{t^5}{6!} + \cdots$$

where in the partial derivatives it is supposed that x and x' are replaced severally by a and a'.

Evidently, for these series proceeding according to ascending powers of t, we may substitute others proceeding according to cosines of multiples of an angle proportional to t, or x and x' may have the form

$$x \text{ or } x' = a_0 + a_1 \cos nt + a_2 \cos 2nt + \ldots.$$

But here we are not entitled to assume that the coefficients rapidly diminish. To bring about such a relation as $n = n'$ it may be necessary for the arbitrary constants we have denoted by e and e' to be of the zero order of magnitude.

If in the expansion of Ω according to powers and products of x and x' we omit all the terms involving both variables the integration of the differential

equations is reduced to quadratures. Let us write in a more general way this expansion of Ω

$$\begin{aligned}\Omega = {} & a_{20}x^2 + a_{11}xx' + a_{02}x'^2 \\ & + a_{30}x^3 + a_{21}x^2x' + a_{12}xx'^2 + a_{03}x'^3 \\ & + a_{40}x^4 + a_{31}x^3x' + a_{22}x^2x'^2 + a_{13}xx'^3 + a_{04}x'^4 \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

It must be noticed that the coefficients of the terms involving both x and x' are of the order of the planetary masses while those which multiply the simple powers of x or x' are of the order of the central mass. If we put

$$\begin{aligned}\Omega_0 = {} & a_{20}x^2 + a_{30}x^3 + a_{40}x^4 + \ldots \\ & + a_{02}x'^2 + a_{03}x'^3 + a_{04}x'^4 + \ldots\end{aligned}$$

the differential equations resulting from this expression of the potential function are

$$\frac{d^2x}{dt^2} = 2a_{20}x + 3a_{30}x^2 + 4a_{40}x^3 + \ldots,$$

$$\frac{d^2x'}{dt^2} = 2a_{02}x' + 3a_{03}x'^2 + 4a_{04}x'^3 + \ldots.$$

From these are immediately derived the integrals

$$\frac{dx^2}{dt^2} = C + a_{20}x^2 + a_{30}x^3 + a_{40}x^4 + \ldots,$$

$$\frac{dx'^2}{dt^2} = C' + a_{02}x'^2 + a_{03}x'^3 + a_{04}x'^4 + \ldots,$$

and thence the final integrals

$$t + c = \int \frac{dx}{\sqrt{(C + a_{20}x^2 + a_{30}x^3 + a_{40}x^4 + \ldots)}},$$

$$t + c' = \int \frac{dx'}{\sqrt{(C' + a_{02}x'^2 + a_{03}x'^3 + a_{04}x'^4 + \ldots)}}.$$

The polynomials under the radical sign, on being equated to 0, will have each two real roots, which, on account of the smallness of x and x', will be, severally, in the neighborhood of the values

$$x = +\sqrt{-\frac{C}{a_{20}}}, \quad x = -\sqrt{-\frac{C}{a_{20}}}, \quad x' = +\sqrt{-\frac{C'}{a_{02}}}, \quad x' = -\sqrt{-\frac{C'}{a_{02}}}.$$

The exact values of these can readily be found by trial; let them be denoted severally, by a, b, a′, b′. Then we can write

$$t + c = \int_{\mathrm{l}}^{x} \frac{dx}{\sqrt{[(\mathrm{a} - x)(x - \mathrm{b})(A_0 + A_1 x + A_2 x^2 + \ldots)]}},$$

$$t + c' = \int_{\mathrm{l}}^{x'} \frac{dx'}{\sqrt{[(\mathrm{a}' - x')(x' - \mathrm{b}')(A'_0 + A'_1 x' + A'_2 x'^2 \ldots)]}}.$$

The limiting values of x are a and b, and those of x', a′ and b′. Within these limits the last factors under the radical sign are always positive and do not vary much, and thus x and x' are periodic functions of t, the periods being given by the formulæ

$$T = 2\int_{\mathrm{l}}^{\mathrm{a}} \frac{dx}{\sqrt{[(\mathrm{a} - x)(x - \mathrm{b})(A_0 + A_1 x + A_2 x^2 + \ldots)]}},$$

$$T' = 2\int_{\mathrm{b}'}^{\mathrm{a}'} \frac{dx'}{\sqrt{[(\mathrm{a}' - x')(x' - \mathrm{b}')(A'_0 + A'_1 x' + A'_2 x'^2 + \ldots)]}}.$$

By adopting, in place of x and x', variables φ and φ' such that

$$2x = (\mathrm{a} + \mathrm{b}) - (\mathrm{a} - \mathrm{b})\cos\varphi, \qquad 2x' = (\mathrm{a}' + \mathrm{b}') - (\mathrm{a}' - \mathrm{b}')\cos\varphi',$$

the integrations are considerably simplified. The motion of φ and φ' is unlimited like that of t. Let

$$[A_0 + A_1 x + A_2 x^2 + \ldots]^{-\frac{1}{2}} = \frac{1}{n}[1 + B_1\cos\varphi + 2B_2\cos 2\varphi + \ldots],$$

$$[A'_0 + A'_1 x' + A'_2 x'^2 + \ldots]^{-\frac{1}{2}} = \frac{1}{n'}[1 + B'_1\cos\varphi' + 2B'_2\cos 2\varphi' + \ldots].$$

Then

$$n(t + c) = \varphi + B_1\sin\varphi + B_2\sin 2\varphi + \ldots,$$

$$n'(t + c') = \varphi' + B'_1\sin\varphi' + B'_2\sin 2\varphi' + \ldots.$$

The values of φ and φ' can be derived from these transcendental equations and have the forms

$$\varphi = n(t + c) + E_1\cos n(t + c) + E_2\cos 2(nt + c) + \ldots,$$

$$\varphi' = n'(t + c') + E'_1\cos n'(t + c') + E'_2\cos 2(n't + c') + \ldots.$$

From these we can pass to the values of $\cos\psi$ and $\cos\psi'$, and thence to those of x and x', which have the form

$$x = \tfrac{1}{2} G_0 + G_1 \cos\varphi + G_2 \cos 2\varphi + \ldots,$$

$$x' = \tfrac{1}{2} G_0' + G_1' \cos\varphi' + G_2' \cos 2\varphi' + \ldots,$$

φ and φ' being put for $nt + c$ and $n't + c'$.

The reversion of the periodic series can be avoided by the use of definite integrals, and we can pass directly to the values of the G_i and G_i'. For we have

$$G_i = \frac{2}{\pi} \int_0^\pi x \cos i\varphi \, d\varphi ,$$

with a similar formula for G_i'. By adopting ψ as the independent variable this becomes

$$G_i = \frac{2}{\pi} \int_0^\pi [(\mathrm{a} + \mathrm{b}) - (\mathrm{a} - \mathrm{b}) \cos\psi] \cos [i\psi + iB_1 \sin\psi + iB_2 \sin 2\psi + \ldots]$$
$$[1 + B_1 \cos\psi + 2B_2 \cos 2\psi + \ldots] \, d\psi .$$

In the particular case of $i = 0$, this reduces to

$$G_0 = \mathrm{a} + \mathrm{b} - \tfrac{1}{2} (\mathrm{a} - \mathrm{b}) \, B_1 .$$

Proposing now to take into account the omitted terms of Ω, which we will denote as Ω_1, let the values of x and x', just determined, be denoted as x_0 and x_0', their corrections as δx and $\delta x'$, the differential equations for the latter will be

$$\frac{d^2 \delta x}{dt^2} = [2a_{20} + 3a_{30}(x + x_0) + 4a_{40}(x^2 + x\,x_0 + x_0^2) + \ldots] \, \delta x + \frac{\partial \Omega_1}{\partial x} ,$$

$$\frac{d^2 \delta x'}{dt^2} = [2a_{02} + 3a_{03}(x' + x'_0) + 4a_{04}(x'^2 + x'\,x'_0 + x_0'^2) + \ldots] \, \delta x' + \frac{\partial \Omega_1}{\partial x'} .$$

These can be integrated by successive approximations, introducing into the right members the values of x and x' from the previous approximation, beginning with $x = x_0$, $x' = x'_0$. In order to prevent t from getting outside of the functional sign cos, we adopt the following device: Let n and n' be two indeterminate constants, whose values come out only at the end of the approximation, but which are equivalent to the rigorous rates of motion of the

arguments φ and φ'. We add $n^2\delta x$ to each member of the first equation and $n'^2\delta x'$ to each member of the second. Then, putting

$$Q = [n^2 + 2a_{20} + 3a_{30}(x + x_0) + 4a_{40}(x^2 + x\,x_0 + x_0^2) + \ldots]\,\delta x + \frac{\partial \Omega_1}{\partial x},$$

and Q' for the similar quantity in the second equation, we have

$$\frac{d^2\delta x}{dt^2} + n^2\delta x = Q\,, \quad \frac{d^2\delta x'}{dt^2} + n'^2\delta x' = Q'\,.$$

In any stage of the approximation we shall have

$$Q = \Sigma K_{i,i'} \cos(i\varphi + i'\varphi')\,, \quad Q' = \Sigma K'_{i,i'} \cos(i\varphi + i'\varphi')\,,$$

from which

$$\delta x = \Sigma . [n^2 - (in + i'n')^2]^{-1} K_{i,i'} \cos(i\varphi + i'\varphi')\,,$$
$$\delta x' = \Sigma . [n'^2 - (in + i'n')^2]^{-1} K'_{i,i'} \cos(i\varphi + i'\varphi')\,.$$

Then the equations for determining n and n' are

$$K_{1,0} = 0\,, \quad K'_{0,1} = 0\,.$$

The terms of Ω in x^2x' and xx'^2 can be removed from it by a linear transformation, but at the expense of reintroducing a term in xx', as also a term involving the product of the velocities into the expression for the living force. For substituting

$$x = y + ly'\,, \quad x' = y' + l'y\,,$$

if l and l' are so determined that

$$3a_{30}l + a_{21}(1 + 2ll') + a_{12}l'(2 + ll') + 3a_{03}l'^2 = 0\,,$$
$$3a_{30}l^2 + a_{21}l(2 + ll') + a_{12}(1 + 2ll') + 3a_{03}l' = 0\,,$$

the terms in Ω involving y^2y' and yy'^2 vanish. And if we put

$$B_{20} = a_{20} + a_{11}l' + a_{02}l'^2\,,$$
$$B_{11} = 2a_{20}l + a_{11}(1 + ll') + 2a_{02}l'\,,$$
$$B_{02} = a_{20}l^2 + a_{11}l + a_{02}\,,$$
$$B_{30} = a_{30} + a_{21}l' + a_{12}l'^2 + a_{03}l'^3\,,$$
$$B_{03} = a_{30}l^3 + a_{21}l^2 + a_{12}l + a_{03}\,,$$

we have

$$\Omega = B_{20}y^2 + B_{11}yy' + B_{02}y'^2 + B_{30}y^3 + B_{03}y'^3 + \ldots\,,$$

and the expression of the living force becomes

$$T = \tfrac{1}{2}(1 + l'^2)\frac{dy^2}{dt^2} + (l + l')\frac{dy}{dt}\frac{dy'}{dt} + \tfrac{1}{2}(1 + l^2)\frac{dy'^2}{dt^2},$$

and the differential equations of the problem are

$$(1 + l'^2)\frac{d^2y}{dt^2} + (l + l')\frac{d^2y'}{dt^2} = 2B_{20}y + 3B_{30}y^2 + B_{11}y',$$

$$(1 + l^2)\frac{d^2y'}{dt^2} + (l + l')\frac{d^2y}{dt^2} = 2B_{02}y' + 3B_{03}y'^2 + B_{11}y.$$

The elimination of l or l' between the equations which determine them gives rise to an equation of the fifth degree, which consequently, always has a real root. As a_{21} and a_{12} are of the order of the masses of the planets, approximate values for l and l' are $l = -\frac{a_{21}}{3a_{30}}$, $l' = -\frac{a_{12}}{3a_{03}}$.

In integrating the preceding equations one may follow either of two courses; first, we may neglect the terms in y^2 and y'^2, which will afterwards be regarded as perturbative, and then we shall have linear equations with constant coefficients easily integrable; or second, we may regard the last terms of both members of each equation as perturbative, and the equations restricted to the remaining terms are integrable by quadratures, as has been shown.

PROOF OF A FORMULA DUE TO CAUCHY.

By Prof. W. H. Echols, Charlottesville, Va.

The function

$$\begin{vmatrix} 1 & , 1 , & 1 , & 1 & , \ldots , & 1 \\ a^p x & , 1 , & a , & a^2 & , \ldots , & a^{n-1} \\ (a^p x)^2 & , 1 , & a^2 , & (a^2)^2 , & \ldots , & (a^{n-1})^2 \\ \cdot & \cdot \cdot & \cdot & \cdot & \cdot \cdot & \cdot \\ (a^p x)^n & , 1 , & a^n , & (a^2)^n , & \ldots , & (a^{n-1})^n \end{vmatrix} \div \zeta^{\frac{1}{2}}(a, a^2 \ldots a^n) , \qquad (1)$$

vanishes when $x = a^{i-p}$ $(i = 0, 1, \ldots, n-1)$, p being any finite quantity.

If we expand (1) with respect to the first column, we have

$$1 + \sum_{r=1}^{r=n} (-1)^r \frac{(1 - a^{-n+r-1}) \ldots (1 - a^{-n})}{(a-1) \ldots (a^r - 1)} a^{r(p+1)} x^r .$$

This rational integral function of x of the nth degree has the n roots a^{-p}, $a^{1-p}, \ldots, a^{n-1-p}$; whence we have the identity

$$\prod_{i=0}^{i=n-1} (1 - a^{p-i}x) = 1 + \sum_{r=1}^{r=n} (-1)^r \frac{(1 - a^{-n+r-1}) \ldots (1 - a^{-n})}{(a-1) \ldots (a^r - 1)} a^{r(p+1)} x^r , \quad (2)$$

Evidently, if $a > 1$, we have

$$\prod_{i=0}^{i=\infty} (1 - a^{p-i}x) = 1 + \sum_{r=1}^{r=\infty} \frac{(-1)^r a^{r(p+1)}}{(a-1) \ldots (a^r - 1)} x^r . \qquad (3)$$

Let $x = a^{q-p}$, where q is any positive integer or zero, then

$$0 = 1 + \sum_{r=1}^{r=\infty} \frac{(-1)^r a^{r(q+1)}}{(a-1) \ldots (a^r - 1)} .$$

In (2) let $p = -1$, we then have Cauchy's formula (Comptes Rendus, 1840. Chrystal's Algebra, II, 316), and (3) becomes Euler's theorem (Introd. in Anal. Inf., § 306).

As a consequence the sum of the products, taken q at a time without repetition, of the quantities

$$a^{-i} \quad (i = 1, 2, 3, \ldots, \infty)$$

is

$$\frac{1}{(a-1) \ldots (a^q - 1)} .$$

ON SOME FORMS OF LAGRANGE'S INTERPOLATION FORMULA.

By Prof. W. H. Echols, Charlottesville, Va.

1. The well known form

$$\frac{\begin{vmatrix} fx & , & fa_1 & , \ldots , & fa_n \\ 1 & , & 1 & , \ldots , & 1 \\ x & , & a_1 & , \ldots , & a_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x^{n-1} & , & a_1^{n-1} & , \ldots , & a_n^{n-1} \end{vmatrix}}{\Sigma^{\frac{1}{2}}(a_1 \ldots a_n)}, \tag{1}$$

which has for its value

$$R = (x - a_1) \ldots (x - a_n) \frac{f^n u}{n!},$$

(u lying between the greatest and least of $x, a_1, \ldots a_n$) may be written out according to the first row, in the standard form

$$fx = \sum_{r=1}^{r=n} fa_r \frac{\Pi (x - a_i)}{\Pi (a_r - a_i)} + R, \quad (i = 1 \ldots n) \text{ Ex } r \tag{2}$$

or by the first row or column according as we give precedence to the Σ sign of r or p, in the formula

$$fx = \sum_{r=1}^{n} \sum_{p=0}^{n-1} (-1)^{r+1} (-1)^{p+1} x^p fa_r P_p \frac{\Pi a_i}{\Pi (a_i - a_r)} + R, \tag{3}$$

wherein $P_0 = 1$, P_p is the sum of the products of the quantities $1/a_i$ taken p at a time, without repetition.

The values of

$$P_p \frac{\Pi a_i}{\Pi (a_i - a_r)},$$

for the three cases $a_r = r$, $a_r = r^2$, $a_r = a^r$, are, respectively,

$$\frac{n!}{r!(n-r)!} P_p, \tag{i}$$

$$2P_p \frac{(1 - 1/n) \ldots (1 - r - 1/n)}{(1 + 1/n) \ldots (1 + r/n)}; \tag{ii}$$

which becomes, when $n = \infty$,

$$2^{r}\mathfrak{F}_p,$$

${}^{r}\mathfrak{F}_p$ being Fourier's number of the rth order and class p.

$$\frac{P_p}{(a-1)\dots(a^r-1)(1-1/a)\dots(1-1/a^{n-r})}, \tag{iii}$$

in which

$$P_p = \sum_{q=0}^{p} \frac{(-1)^q}{a^{rq}(a-1)\dots(a^{p-q}-1)}$$

$$= S_p - \frac{1}{a^r} P_{p-1},$$

$$S_p \quad (a-1)\dots(a^p-1).$$

The corresponding values of R are in these cases

$$R = (-1)^n (1-x/1)\dots(1-x/n) \Delta^n u, \tag{i}$$

$$R = (-1)^n (1-x/1^2)\dots(1-x/n^2)\, n! \Delta^n u, \tag{ii}$$

$$R = (-1)^n (1-x/a)\dots(1-x/a^n)\, a^{\frac{1}{2}n(n+1)} \Delta^n u/n! \tag{iii}$$

2. Instead of writing the determinant (1) in solid columns, we may write it

$$\frac{\begin{vmatrix} fx & , x^{n1} & , x^{n2} & , \dots, & x^{nn} \\ fa_0 & , a_0^{n1} & , a_0^{n2} & , \dots, & a_0^{nn} \\ fa_1 & , 0 & , a_1^{n2} & , \dots, & a_1^{nn} \\ \dots & \dots & \dots & \dots & \dots \\ fa_{n-1} & , 0 & , 0 & , \dots, & a_{n-1}^{nn} \end{vmatrix}}{a_0^{n1} \dots a_{n-1}^{nn}}, \tag{4}$$

wherein

$$x^{nr} = (x-a_n)\dots(x-a_r).$$

(4) gives

$$fx = \sum_{p=0}^{n-1} A_p fa_p + \frac{x^{n0}}{(n+1)!} \Delta^n u. \tag{5}$$

in which we have for computing the A's, $A_0 = x^{n1}/a_0^{n1}$, and

$$(-1)^p a_p^{n(p+1)} A_p = a_{p-1}^{n(p+1)} A_{p-1} - a_{p-2}^{n(p+1)} A_{p-2} + \dots + (-1)^{p-1} a_0^{n(p+1)} A_0 + (-1)^p x^{n(p+1)}.$$

3. Again, we have

$$\frac{\begin{vmatrix} fx & , x^{10} , & x^{20} , \ldots , & x^{(n-1)0} \\ fa_1 & , a_1^{10} , & 0 , \ldots , & 0 \\ fa_2 & , a_2^{10} , & a_2^{20} , \ldots , & 0 \\ \cdot & \cdot \cdot \cdot & \cdot \cdot \cdot & \cdot \cdot \cdot \\ fa_{n-1} & , a_{n-1}^{10} , & a_{n-1}^{20} , \ldots , & a_{n-1}^{(n-1)0} \end{vmatrix}}{a_1^{10} \ldots a_{n-1}^{(n-1)0}} , \tag{6}$$

wherein

$$x^{n0} = (x - a_n) \ldots (x - a_0) .$$

(6) gives

$$fx = \sum_{r=1}^{n-1} A_r x^{r0} + \frac{x^{n0}}{n!} f^n u , \tag{7}$$

in which we have for computing the A's, $A_1 = fa_1/a_1^{10}$, and

$$(-1)^{r+1} a_r^{(r-1)0} A_r = a_r^{(r-1)0} A_{r-1} - a_r^{(r-2)0} A_{r-2} + \ldots + (-1)^r a_r^{10} A_1 + (-1)^{r+1} fa_r .$$

CHARLOTTESVILLE, *July*, 1893.

SOLUTIONS.

330

FIND the sum of the series

$$1^2 + 3^2 + 6^2 + 10^2 + 15^2 + \ldots + [\tfrac{1}{2} n (n+1)]^2.$$

[*Artemas Martin.*]

SOLUTION.

The fifth differences vanish and the first terms of the difference series are 1, 8, 19, 18, 6, 0; hence

$$S = {}_nC_1 + {}_nC_2 \cdot 8 + {}_nC_3 \cdot 19 + {}_nC_4 \cdot 18 + {}_nC_5 \cdot 6,$$

where ${}_nC_1$, etc. are the binomial coefficients. [*Geo. R. Dean.*]

332

FOUR equianharmonic points give four triangles which have four circumcircles. Show that the inverses of any point with regard to these four circles are equianharmonic. [*Frank Morley.*]

SOLUTION.

It is proved in Salmon's Conic Sections, § 54, that if the perpendiculars from three points a_1, a_2, a_3 on the sides of a triangle $c_1c_2c_3$ meet at a point a_4, then also the perpendiculars from c_1, c_2, c_3 on the sides of $a_1a_2a_3$ meet at a point c_4. And this can be readily proved by elementary geometry, by observing that the condition of either concurrence is, that in the hexagon a_1, c_3, a_2, c_1, a_3, c_2 the sum of the squares of three alternate sides is equal to the sum of the squares of the other three sides.

Now

$$\left| \frac{c_2 - c_3}{c_1 - c_3} \right| = \frac{\sin c_2c_1c_3}{\sin c_3c_2c_1} = \frac{\sin a_2a_4a_3}{\sin a_3a_4a_1},$$

$$\left| \frac{c_1 - c_4}{c_2 - c_4} \right| = \frac{\sin c_4c_2c_1}{\sin c_4c_1c_2} = \frac{\sin a_1a_3a_4}{\sin a_4a_3a_2};$$

whence

$$\left| \frac{c_2 - c_3 \,.\, c_1 - c_4}{c_1 - c_3 \,.\, c_2 - c_4} \right| = \frac{\sin a_2a_4a_3 \,.\, \sin a_1a_3a_4}{\sin a_4a_3a_2 \,.\, \sin a_3a_4a_1} = \left| \frac{a_2 - a_3 \,.\, a_1 - a_4}{a_2 - a_4 \,.\, a_1 - a_3} \right|.$$

Hence two corresponding cross-ratios of the two tetrads of points have equal

absolute values. They have also congruent amplitudes ; for

$$\text{am}\,(c_2 - c_3) \quad \text{am}\,(a_1 - a_4) - \pi/2\,,$$

$$\text{am}\,(c_1 - c_4) \quad \text{am}\,(a_2 - a_3) + \pi/2\,;$$

and, therefore,

$$\text{am}\,(c_2 - c_3\,.\,c_1 - c_4) \quad \text{am}\,(a_2 - a_3\,.\,a_1 - a_4)\,.$$

Hence corresponding cross-ratios are equal.

In particular, let c_1, c_2, c_3 be the centres of the circles $a_2a_3a_4$, $a_3a_4a_1$, $a_4a_1a_2$; then c_4 is the centre of the circle $a_1a_2a_3$; and the theorem is: The inverses of the point ∞ , with regard to the four circles through three of any four points a, have the same cross-ratios as the four points. This is a covariantive statement, in which we can substitute any other point for the point ∞ .

[*Frank Morley.*]

339

Show that the areas of the curves

$$y = px^2 + qx + r\,, \text{ and } y = mx^3 + px^2 + qx + r\,,$$

taken between the limits $x + h$ and $x - h$, are given by the formulæ

$$\Omega = 2yh + \tfrac{2}{3}\,ph^3\,, \text{ and } \Omega = 2yh + (\tfrac{2}{3}\,p + 2mx)h^3\,,$$

respectively. [*W. H. Echols.*]

SOLUTION I.

From the well-known formula

$$\Omega = \int y\,dx$$

we have for the first curve

$$\Omega = \int_{x-h}^{x+h} (px^2 + qx + r)\,dx = \tfrac{1}{3}\,px^3 + \tfrac{1}{2}\,qx^2 + rx\Big]_{x-h}^{x+h}$$

$$= 2h\,(px^2 + qx + r) + \tfrac{2}{3}\,ph^3 = 2yh + \tfrac{2}{3}\,ph^3\,;$$

and for the second curve

$$\Omega = \int_{x-h}^{x+h} (mx^3 + px^2 + qx + r)\,dx = \tfrac{1}{4}\,mx^4 + \tfrac{1}{3}\,px^3 + \tfrac{1}{2}\,qx^2 + rx\Big]_{x-h}^{x+h}$$

$$= 2\text{h}\,(mx^3 + px^2 + qx + r) + (\tfrac{2}{3}\,p + 2mx)\,h^3 = 2yh + (\tfrac{2}{3}\,p + 2mx)\,h^3\,.$$

[*Artemas Martin.*]

SOLUTION II.

By the familiar formula called Simpson's rule the mean value of

$$y = mx^3 + px^2 + qx + r$$

is the sixth part of the sum of the extreme values plus four times the middle value. But

$$(x+h)^3 + (x-h)^3 + 4x^3 = 6x^3 + 6h^2x\,,$$
$$(x+h)^2 + (x-h)^2 + 4x^2 = 6x^2 + 2h^2\,,$$
$$(x+h) + (x-h) + 4x = 6x\,;$$

$$\therefore \qquad y_m = mx^3 + px^2 + qx + r + mh^2x + \tfrac{1}{3}ph^2$$
$$= y + h^2(mx + \tfrac{1}{3}p)\,,$$

and
$$A = 2hy + h^3(2mx + \tfrac{2}{3}p)\,.$$

Putting $m = 0$, we have the other result. [*W. M. Thornton.*]

SOLUTION III.

As applications of

$$\tfrac{1}{2}\int_{x-h}^{x+h} fx\,.\,dx = hfx + \frac{h^3}{3!}f''x + \frac{h^5}{5!}f^{\text{IV}}x + \ldots\,,$$

we have for the first one,

$$\tfrac{1}{2}\,\Omega = hy + \frac{h^3}{3!}2p\,;$$

for the second

$$\tfrac{1}{2}\,\Omega = hy + \frac{h^3}{3!}(3\,.\,2mx + 2p)\,. \qquad [\textit{W. H. Echols.}]$$

346

A WOODEN hemisphere floats in water, vertex down, with $1/n$ of its axis immersed. Find the specific gravity of the hemisphere.

[*Artemas Martin.*]

SOLUTION.

Let s be the specific gravity of the hemisphere and r its radius. The weight of the hemisphere equals the weight of the water displaced, or,

$$\tfrac{2}{3}\pi r^3 s = \pi\left[\frac{r}{n}\right]^2\left[r - \tfrac{1}{3}\,.\,\frac{r}{n}\right],$$

which gives

$$s = \frac{3n-1}{2n^3}\,. \qquad [\textit{W. O. Whitescarver.}]$$

352

A SOLID sphere and a solid cylinder of equal radii roll from rest down the same inclined plane; compare the times of their descent.

[*Artemas Martin.*]

SOLUTION.

By the principle of vis viva,

$$mv^2 + I\omega^2 = 2mgs \sin i .$$

The motion being pure rolling,

$$v = r\omega .$$

For the sphere,

$$I = \tfrac{2}{5}\, mr^2 .$$

For the cylinder,

$$I = \tfrac{1}{2}\, mr^2 .$$

Substituting these values and reducing, we have, for the sphere,

$$v = \frac{ds}{dt} = \sqrt{\tfrac{5}{14}\, gs \sin i} ;$$

and for the cylinder,

$$v = \frac{ds}{dt} . = \sqrt{\tfrac{1}{3}\, gs \sin i} .$$

Integrating, we get

$$\sqrt{s} = t \sqrt{\tfrac{5}{14}\, g \sin i} , \qquad \sqrt{s} = t' \sqrt{\tfrac{1}{3}\, g \sin i} ;$$

whence

$$\frac{t}{t'} = \sqrt{\tfrac{14}{15}} .$$

[*S. T. Moreland; M. C. Andrews; C. E. Mendenhall; G. R. Dean.*]

EXERCISE.

357

GIVEN $\tan_\kappa w = x + iy$, wherein

$$\tan_\kappa w = \kappa (e^{w/\kappa} - e^{-w/\kappa})/(e^{w/\kappa} + e^{-w/\kappa}) ,$$

$$w = u + iv , \quad i = \sqrt{-1} ,$$

and

$$x = m (\cos \beta + i \sin \beta) ;$$

determine u and v as real quantities in terms of x, y, m, and β.

[*Irving Stringham.*]